it is necessary that two waves radiate from the discontinuity. This occurs when the velocity upstream of the discontinuity is higher than the speed of sound, while the velocity downstream of it is lower than the speed of sound, i.e. $v_1 > a_1$ and $v_2 < a_2$. The condition $v_1 > a_1$ implies that

$$v_1^2 > a_0^2 + \alpha^2\rho_1 E_1^2 / 4\pi\varepsilon \quad \text{or} \quad -\Omega + e^2\alpha^{*2} / (\alpha^* + 1) < -\gamma \qquad (3.1)$$

The left-hand side of the last inequality contains a quantity which is the tangent of the slope of the curve which represents the equation of conservation of momenta, while that in the right-hand side is the tangent of the shock adiabate at point $V = 1$, $P = 1$. It follows from (3.1) that in the case of evolutionary waves the line corresponding to the equation of conservation of momenta at point $V = 1$, $P = 1$ for $V < 1$ lies above the shock adiabate and must always intersect the latter in the interval $1 / k < V < 1$. We have thus established that in electrohydrodynamics in the case of linear dependence of permittivity on density shock waves are always compression waves. The normal component of the elctric field downstream of the wave front is smaller than the normal component of that field upstream of the front.

It will be seen from (2.1) that for $V < 1$ the curve of the shock adiabate lies higher than the conventional gasdynamic adiabate, i.e. it is in region $S > S_1$, where S_1 is the entropy at point $P = 1$, $V = 1$. The increase of entropy at the shock also shows that the latter is a compression shock.

REFERENCES

1. Kulikovskii, A.G and Liubimov, F.A., Magnetohydrodynamics. Fizmatgiz, Moscow, 1962.
2. Landau, L.D. and Lifshits, E.M., Electrodynamics of Continuous Media. (English translation), Pergamon Press, Book № 09105, 1960.
3. Sedov, L.I., Mechanics of Continuous Medium. "Nauka", Moscow, 1970. (See also Sedov, L.I. Foundations of Nonlinear Mechanics of Continua, Pergamon Press, Book № 09878, 1965).

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ON THE STABILITY OF COUETTE FLOW OF A SECOND-ORDER FLUID

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The stability of a Couette flow of incompressible non-Newtonian second-order fluid at high Reynolds numbers [1] is considered within the limits of the linear theory of hydrodynamic stability. Unlike the Couette flow of a Newtonian (first-order) fluid which according to the linear theory is stable, the flow considered here may loose its stability even in the linear approximation.

The problem of hydrodynamic stability of simple flows of non-Newtonian fluids was considered in a fairly large numer of publications [2-4] in which the effect of elastic properties of non-Newtonian fluids on stability was examined. Stability fluctuations induced by small deviations from Newtonian properties in unstable and in conventional Newtonian fluid flows were analyzed in [2, 3]. The stability of a plane-parallel Couette flow was investigated in [4] with the use of the linear theory of stability, where the destabilizing effect of pronounced elastic properties of fluid at low Reynolds numbers on the flow was pointed out.

The problem of stability of the Couette flow at high Reynolds numbers is investigated here in the case of an inelastic second-order fluid whose viscous stress tensor is defined by (*)

$$\sigma_{ij} = \rho\nu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) - a\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)\left(\frac{\partial v_k}{\partial x_j} + \frac{\partial v_j}{\partial x_k}\right) + b\left(\frac{\partial v_i}{\partial x_k}\frac{\partial v_j}{\partial x_k} - \frac{\partial v_k}{\partial x_i}\frac{\partial v_k}{\partial x_j}\right) + \varkappa\left(\frac{\partial}{\partial t} + v_k\frac{\partial}{\partial x_k}\right)\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

where ν is the coefficient of kinematic viscosity, ρ is the fluid density, and a, b and $\varkappa$ are coefficients of normal stresses.

1. The stream function ψ of a plane isothermic flow of an incompressible viscous second-order fluid satisfies the equation

$$\frac{\partial\Delta\psi}{\partial t} + \frac{\partial(\Delta\psi, \psi)}{\partial(x, y)} = \nu\Delta\Delta\psi + \frac{b+\varkappa}{\rho}\Delta\frac{\partial(\Delta\psi, \psi)}{\partial(x, y)} - \frac{b}{\rho}\frac{\partial(\Delta\Delta\psi, \psi)}{\partial(x, y)} + \frac{\varkappa}{\rho}\frac{\partial}{\partial t}\Delta\Delta\psi \tag{1.1}$$

Let us consider the stability of a flow whose velocity profile $U(y)$ with respect to small perturbations of the stream function ψ' is of the form

$$\psi' = f(y)\, e^{i\alpha(x-ct)} \tag{1.2}$$

Linearizing Eq. (1.1) and using (1.2), for $f(y)$ in dimensionless coordinates we obtain

$$[U - c + (\beta+\gamma)k\alpha^2 U]D^2 f - [1 + \alpha^2 k(\beta+\gamma)]U''f = \tag{1.3}$$

$$-\left(k\beta U + ck\gamma + \frac{i}{\alpha R}\right)D^2D^2 f + k\beta f U^{\mathrm{IV}} + k(\beta+\gamma)(UD^2 f - U''f)'', \quad D^2 = d^2/dy^2 - \alpha^2$$

$$\beta = \frac{b}{m\rho\nu^2/T}, \quad \gamma = \frac{\varkappa}{m\rho\nu^2/T}, \quad k = \frac{m\nu^2}{TL^2}, \quad R = \frac{u_0 L}{\nu}$$

where the dimensionless constants β and γ (of the order of unity) depend on the selection of a particular statistical model for the fluid, m is the mass of a fluid molecule, T is the fluid temperature measured in energy units ($[T]$ = erg), k is a dimensionless parameter which defines the ratio of non-Newtonian to inertial forces in the fluid, L is a characteristic dimension, and u_0 is the flow velocity.

Let us consider the stability of the Couette flow $U = y$ with respect to perturbations which satisfy the condition $0 < \alpha \sim 1$. We set $\beta = 0$ in Eq. (1.3), which is valid for some statistical models of fluid. Then for high Reynolds numbers and $k \gg 1/R$ the equation of perturbations becomes

$$(y - c + \gamma k\alpha^2 y)D^2 f = -ck\gamma D^2D^2 f + \gamma k(yD^2 f)'' \tag{1.4}$$

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*) Savchenko, V. A., Candidate's dissertation, Rostov-on-Don, 1972.

A transition to limit $R \to \infty$ does not in this case correspond to any particular perturbations [5], since terms containing higher derivatives do not vanish.

At the channel walls $y = \pm {}^1/_2$ perturbations of the longitudinal and transverse velocities vanish, hence

$$f(\pm {}^1/_2) = f'(\pm {}^1/_2) = 0 \tag{1.5}$$

2. The general solution of the linear equation (1.4) can be written in the form of linearly independent solutions $f_i (i = 1, \ldots, 4)$

$$f = \sum_{i=1}^{4} c_i f_i(y), \qquad c_i = \text{const} \tag{2.1}$$

Owing to the homogeneity of boundary conditions (1.5) the solution (2.1) is nontrivial, if the determinant

$$| f_i({}^1/_2)\ f_i'({}^1/_2)\ f_i(-{}^1/_2)\ f_i'(-{}^1/_2) | = 0 \tag{2.2}$$

The first pair of exact nonlinear solutions is of the simple form

$$f_1(y) = e^{\alpha y}, \qquad f_2(y) = e^{-\alpha y} \tag{2.3}$$

Let us consider flows which satisfy the condition $k \ll 1$ and seek the second pair of solutions in the form of the asymptotic series

$$f = f_0(\eta) + \Delta_1(\gamma k) f_1(\eta) + \ldots \quad \eta = (y - c) / \Delta(\gamma k), \qquad \Delta,\ \Delta_1 = 0(1) \tag{2.4}$$

Note that c, and consequently η can be complex. Substituting (2.4) into (1.4) and using the principle of minimum degeneration of equations [5], in the zero approximation we have

$$\Delta(\gamma k) = \sqrt{\gamma k}, \qquad f_0^{\mathrm{IV}} + \frac{2}{\eta} f_0''' - f_0'' = 0 \tag{2.5}$$

The fundamental system of linearly independent solutions of Eq. (2.5) is of the form

$$f_0^{(1)} = 1, \quad f_0^{(2)} = \eta, \quad f_0^{(3)} = \eta Ei(\eta) - e^{\eta} \tag{2.6}$$

$$f_0^{(4)} = \eta Ei(-\eta) + e^{-\eta}, \qquad Ei(\eta) = \int_{-\infty}^{\eta} e^t \frac{dt}{t}$$

As the second pair of solutions to supplement the pair (2.3) it is necessary to take $f_0^{(3)}$ and $f_0^{(4)}$, since solutions $f_0^{(1)}$ and $f_0^{(2)}$ are linear combinations of expansions of solutions f_1 and f_2 for $|y - c| \ll 1$. Solutions $f_0^{(3)}$ and $f_0^{(4)}$ represent multiple-valued functions with a transcendental branching point for $\eta = 0$. For these solutions to have regular asymptotics for $|y - c| \gg \sqrt{\gamma k}$, i.e.

$$f_3 \simeq \frac{1}{y - c} \exp\left(\frac{y}{\sqrt{\gamma k}}\right), \qquad f_4 \simeq \frac{1}{y - c} \exp\left(-\frac{y}{\sqrt{\gamma k}}\right)$$

it is necessary to separate the required branch of the integral exponential function. For this it is necessary to make a cut along the negative imaginary semiaxis in the plane of the complex variable η, which means that η must satisfy the condition

$$-\pi / 2 < \arg \eta < 3\pi / 2 \tag{2.7}$$

Equalities (2.3) and (2.6) yield the required system of linearly independent solution of Eq. (1.4), which is then used in the characteristic equation (2.2).

3. Let us prove that the considered mode of the Couette flow is unstable. For this we shall consider the case of

$$c_r = \mathrm{Re}\, c < 0, \qquad |c_i| = |\mathrm{Im} c| \sim \sqrt{\gamma k}$$

It is seen that the first two rows of the determinant (2.2) are of the order of unity. Then, depending on which of solutions $f_0^{(3)}$ or $f_0^{(4)}$ is specified as bounded for $y \in [-1/2, 1/2]$ and $k \to 0$, either $f_0^{(3)}$ or $f_0^{(4)}$ will be present in the characteristic equation, and there are two possible waves. Let us consider the one which appears when boundedness of $f_0^{(4)}$ is specified. This requirement is equivalent to the fulfilment of the following relationship:

$$c_r = -\ 1/2 + \theta \sqrt{\gamma k}, \qquad 0 < \theta \sim 1 \tag{3.1}$$

and it also corresponds to the condition $c_r < 0$. Retaining in the determinant (2.2) the term of the highest order of magnitude, we obtain

$$f_3'(1/2)\ f_4'(-1/2)\ \mathrm{sh}\,\alpha = 0 \tag{3.2}$$

Only the second of the three cofactors in the left-hand part can vanish, while the first by virtue of condition (3.1) is equal to $\sqrt{\gamma k}\exp(1/\sqrt{\gamma k})\,[1 + O(\sqrt{\gamma k})]$ and for $k \to 0$ tends to ∞. Hence, as previously stated, the characterisitc equation has only a limited solution. With the use of solutions (2.6), we obtain

$$f_4'\left(-\frac{1}{2}\right) \sim E_i\left(\frac{1/2 + c}{\sqrt{\gamma k}}\right) = 0 \tag{3.3}$$

Only the roots of (3.3) which by virtue of condition (3.1) satisfy the inequality

$$\mathrm{Re}\,[(1/2 + c)/\sqrt{\gamma k}] > 0 \tag{3.4}$$

are of interest.

Note that, since Eq. (3.3) has no real roots, the considered flow with the restrictions defined above does not allow indifferent oscillations. Equation (3.3) has the following complex root:

$$(1/2 + c)\ /\ \sqrt{\gamma k} \simeq 1.31\ e^{0.413\pi i}$$

which satisfies the specified conditions. Thus the real and imaginary parts of c are, respectively, defined by the equalities

$$c_r \simeq -1/2 + 0.31\ \sqrt{\gamma k} + \ldots, \qquad c_i \simeq 1.1\ \sqrt{\gamma k} + \ldots$$

A positive c_i implies that the considered wave is of the intensifying kind. Hence the Couette flow of second-order fluid admits for $R \to \infty$ increasing wave oscillations and is unstable (unlike the Couette flow of a Newtonian fluid which according to the linear theory remains stable). Note that the condition $1 \gg k \gg 1/R$ imposed above means that in Eq. (1.4) the non-Newtonian properties of the fluid ($k \gg 1/\mathrm{Re}$) play a decisive part and that the order of magnitude of inertial terms is higher than that of non-Newtonian ($1 \gg k$).

We note in conclusion that $c_i \to 0$ when $k \to 0$, which shows that with decreasing effect of non-Newtonian properties of the fluid the instability tends to vanish. Results of this investigation show that the destabilizing effect is not only due to the elastic properties of the fluid, as inducated in [4], but also to its non-Newtonian properties, if terms which are quadratic with respect to velocity are taken into account in the expression for the viscous stress tensor.

REFERENCES

1. Savchenko, V.A. and Khazanovich, T.N., Equations of motion of second-order fluid. TMF, Vol. 4, №2, 1970.

2. Thomas, R. H. and Walters, K., The stability of elastico-viscous flow between rotating cylinders, pt. 1. J. Fluid Mech., Vol. 18, №1, 1964.
3. Listrov, A. T., On the stability of flow of a viscoelastic fluid drained onto an inclined plane. PMTF, №5, 1965.
4. Gorodtsov, V. A. and Leonov, A. I., On a linear instability of a plane-parallel Couette flow of viscoelastic fluid. PMM Vol. 31, №2, 1967.
5. Van Dyke, M. D., Perturbation Methods in Fluid Mechanics. Acad. Press, New York - London, 1964.

Translated by J. J. D.

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ON THE PLANE PROBLEM OF THE THEORY OF ELASTICITY FOR MULTIPLY-CONNECTED DOMAINS WITH CYCLIC SYMMETRY

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We consider the problem of determining the stresses in a thin, homogeneous disc, weakened by N like holes situated at the same distance from the center and acted upon by a constant normal load applied along its periphery. Such a cyclically symmetric problem was solved by Buivol in [1], who reduced the Sherman integral equation [2, 3] along the boundary L of the region in question, to an equation along the part of L designated by l and lying within the angle $\theta_0 \leqslant \theta \leqslant \theta_0 + \tau$, where $\tau = 2\pi / N$ and θ is the angular coordinate of the points of l in the polar coordinate system chosen in the plane of the annulus in the usual manner, and θ_0 is arbitrary.

Such an approach utilizes the symmetry of the problem when the resulting equations are solved numerically and, unlike other methods [4-6], it does not impose any restrictions on the size and distribution of the holes, while a suitable choice of the norm in the method of least squares ensures uniform convergence of the complex potentials $\varphi(z)$ and $\psi(z)$ and their derivatives right up to their boundaries. Unfortunately, the paper [1] contains an error. The transformation of the function $\omega(t)$ under a rotation by the angle τ is determined with the accuracy of only up to its principal term, i.e. up to the limiting value of the function holomorphic outside the region in question (see [7] for a representation of holomorphic functions in terms of the Cauchy integrals). Such a limiting value affects the form of $\psi(z)$ and hence the result. In the present paper this value is determined with help of the condition of transformation of $\psi(z)$ under rotation, used in [1].

It is proved that $\omega(t)$ belongs to some subspace $W_2^3(L, \tau)$ of the space $W_2^3(L)$, constructed by taking into account the symmetry of the problem. The application of the method of least squares in $W_2^3(L, \tau)$ leads to an economic computational scheme. We give numerical results for $N = 4$ in the case of different disk-geometries. The method of solution can be easily extended to the